Feynman Lectures on Physics
Lecture 41: The Brownian Movement

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Desiderata

Applications of Equipartition Theorem

Equipartition Theorem

In thermal equilibrium, a classical system whose energy is the sum of n quadratic modes (or degrees of freedom) has a mean energy given by $n \times \frac{1}{2}kT$.

Examples:

- Mass attached to a spring has a mean energy, $2 \times \frac{1}{2}kT = kT$
- Monoatomic gas molecule has a mean energy, $3 \times \frac{1}{2}kT = \frac{3}{2}kT$

Limitations:

- Valid only at high temperature, so that discrete nature of energy can be ignored
- Valid in the limit where harmonic approximation holds true

Applications of Equipartition Theorem

Equipartition Theorem

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Light-Beam Galvanometer

Thermal fluctuations set limits on sensitivity of electrical instruments.

$$\frac{1}{2} I\omega_0^2 \langle \theta^2 \rangle = \frac{1}{2} kT$$

So, $\langle \theta^2 \rangle = \frac{kT}{I\omega_0^2}$ is the thermal fluctuation in galvanometer at temperature T.

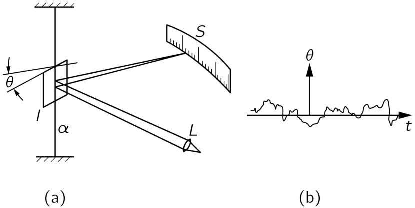


Fig. 41–1. (a) A sensitive light-beam galvanometer. Light from a source L is reflected from a small mirror onto a scale. (b) A schematic record of the reading of the scale as a function of the time.

Resonant Circuit-Johnson Noise

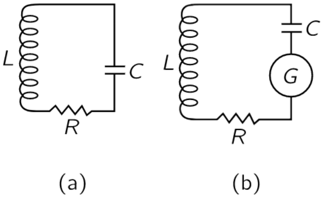
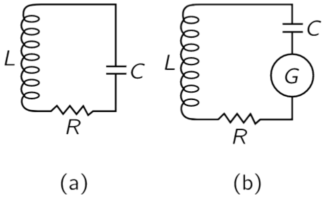


Fig. 41–2. A high-Q resonant circuit. (a) Actual circuit, at temperature T. (b) Artificial circuit, with an ideal (noiseless) resistance and a "noise generator" G.

What is the fluctuations in the voltage of the inductor?

Mean energy of inductor $\langle E \rangle = \frac{L\langle I^2 \rangle}{2} = \frac{1}{2}kT$. Voltage across inductor, $V = i\omega_0 LI$. So, $\langle V^2 \rangle = \omega_0^2 LkT$. This is called Johnson noise.

Resonant Circuit-Johnson Noise



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Blackbody radiation

Imagine gas of atoms.

- At temperature T, electrons inside the atom will be oscillating with energy kT (in 1 dimension).
- Since, electrons are charged it'll radiate light into the environment lowering its temperature.

Imagine that there are mirrors on the walls so that the radiated light can be reflected back and scatter the electrons again.

Question

What will be the energy spectra of the radiated light at thermal equilibrium? This light is called the blackbody radiation.

We solve this problem by balancing light radiated and absorbed by the oscillators per unit time.

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Energy radiated: Classical Oscillator

We assume that electrons oscillate in an atom under harmonic oscillator potential. We begin by defining quality factor(Q) for the oscillator:

$$Q = \frac{\text{energy of oscillator}}{\text{energy radiated per radian}} = \frac{W}{dW/d\phi} = \frac{\omega_0 W}{dW/dt}$$

Q for an atom is given by, $Q = \frac{3c}{2r_0\omega_0} \sim 10^8$. Here, r_0 is classical electron radius. So,

$$\frac{dW}{dt} = \frac{\omega_0 W}{Q} = \frac{2r_0\omega_0^2 W}{3c}$$

Averaging over-time and using equipartition theorem in three dimensions gives,

$$\langle\frac{dW}{dt}\rangle = \frac{2}{3}\frac{r_0\omega_0^2\langle W\rangle}{c} = \frac{2}{3}\frac{r_0\omega_0^2(3kT)}{c} = 3\gamma kT$$

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Energy absorbed by the oscillators

Let $I(\omega)$ be the spectral distribution of light which gives the light energy density at frequency ω. Our goal is to find $I(\omega)$. The cross-section of the oscillators at frequency ω is given by,

$$\sigma_s = \frac{8\pi r_0^2}{3} \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + \gamma^2\omega^2}$$

Since, Q is large we can approximate it as follows,

$$\sigma_s = \frac{2\pi r_0^2 \omega_0^2}{3[(\omega - \omega_0)^2 + \gamma^2/4]}$$

Energy absorbed by the oscillators per unit time is given by,

$$\frac{dW_s}{dt} = \int_0^\infty I(\omega)\sigma_s(\omega)d\omega = \int_0^\infty \frac{2\pi r_0^2 \omega_0^2 I(\omega)}{3[(\omega - \omega_0)^2 + \gamma^2/4]} d\omega$$

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Classical Oscillator

Since, $\sigma_s(\omega)$ is narrowly peaked, we can assume $I(\omega)$ is nearly constant and bring it out of the integral and equate to energy absorption rate.

$$\frac{dW_s}{dt} = \frac{2\pi r_0^2 \omega_0^2 I(\omega_0)}{3} \int_0^\infty \frac{1}{[(\omega - \omega_0)^2 + \gamma^2/4]} d\omega = 3\gamma kT$$

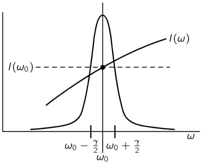

Fig. 41–3. The factors in the integrand (41.10). The peak is the resonance curve $1/[(\omega-\omega_0)^2+\gamma^2/4]$. To a good approximation the factor $I(\omega)$ can be replaced by $I(\omega_0)$.

Rayleigh's law: Ultraviolet Catastrophe

Since our expression for $I(\omega_0)$ is valid for any ω

$$I(\omega) = \frac{\omega^2 kT}{\pi^2 c^2}$$

This is called Rayleigh's law

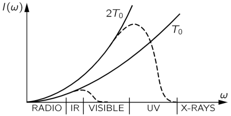

Fig. 41–4. The blackbody intensity distribution at two temperatures, according to classical physics (solid curves). The dashed curves show the actual distribution.

Quantum Oscillator

Planck's assumption

Harmonic oscillator can oscillate only with energy $n\hbar\omega$ where n is non-negative integer. The idea that they can have any energy is false.

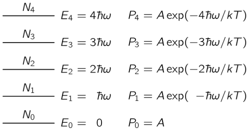

Fig. 41–5. The energy levels of a harmonic oscillator are equally spaced: $E_n = n\hbar\omega$.

Probability of occupying energy level E is given by, $P(E) \propto e^{-E/kT} = e^{-\hbar\omega/kT} = x$. Goal is to find the average energy of the oscillator at temperature T.

Quantum Oscillator

Let there be N_0 oscillators in ground state $E_0 = 0$.
Total number of oscillators is, $N_{tot} = N_0 + N_0 x + N_0 x^2 + \ldots$.
Total energy of the oscillator is, $E_{tot} = N_0 \hbar\omega(0 + 1x + 2x^2 + \ldots)$
Average energy of oscillator is then,

$$\langle E \rangle = \frac{E_{tot}}{N_{tot}} = \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1}$$

Replacing kT with $\langle E \rangle$ gives,

$$I(\omega)d\omega = \frac{\hbar\omega^3}{\pi^2 c^2 (e^{\frac{\hbar\omega}{kT}} - 1)} d\omega$$

Johnson noise-revisited

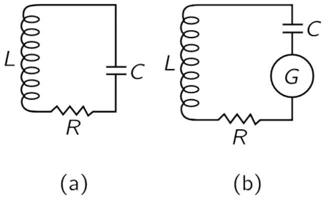

Fig. 41–2. A high-Q resonant circuit. (a) Actual circuit, at temperature T. (b) Artificial circuit, with an ideal (noiseless) resistance and a "noise generator" G.

Using Kirchoff's law,

$$V_R + V_C + V_L = V_G$$

$$\Rightarrow IR + \frac{I}{j\omega C} + Ij\omega L = V_G$$

Solving for I gives,

$$I = \frac{V_G}{R + j(\omega L - 1/(\omega C))}$$

Johnson noise-revisited

Power generated to the resistor by generator G is given by,

$$P = \int_0^\infty |I|^2 R\, d\omega = \int_0^\infty \frac{V_G^2}{R^2 + (\omega L - 1/(\omega C)^2} R\, d\omega$$

Since, R is small, we have a high Q circuit and we can take $V_G(\omega)$ out of the integral to get,

$$P = V_G^2 R \int_0^\infty \frac{1}{R^2 + (\omega L - 1/(\omega C)^2} d\omega = V_G^2 R \frac{\pi}{2RL} = V_G^2 \frac{\pi}{2L}$$

Equating to $\gamma k T d\omega$. But since, $Q = \omega_0 L/R \implies \gamma = R/L$. So, we get,

$$P(\omega) = \frac{V_G^2}{R} = \frac{2kT}{\pi} d\omega$$

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Random Walk

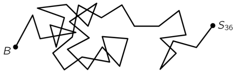

Fig. 41–6. A random walk of 36 steps of length l. How far is S_{36} from B? *Ans:* about $6l$ on the average.

$\vec{R}_N = \vec{R}_{N-1} + \vec{L}$, squaring it we get,

$$\vec{R}_N \cdot \vec{R}_N = \vec{R}_N^2 = \vec{R}_{N-1}^2 + 2\vec{R}_{N-1} \cdot L + L^2$$

$$\langle \vec{R}_N^2 \rangle = NL^2$$

Random Walk

$$m\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} = F_{\text{ext}}$$

Multiply by x and take time average,

$$m\langle x\frac{d^2x}{dt^2}\rangle + \mu\langle x\frac{dx}{dt}\rangle = \langle xF_{\text{ext}}\rangle$$

$$-\langle mv^2\rangle + \frac{\mu}{2}\frac{d}{dt}\langle x^2\rangle = 0$$

$$\frac{d\langle x^2\rangle}{dt} = 2\frac{kT}{\mu}$$

But $\langle R^2\rangle = \langle x^2\rangle + \langle y^2\rangle + \langle z^2\rangle$.This gives,

$$\langle R^2\rangle = 6kT\frac{t}{\mu}$$